

INFINITE DIMENSION OF SOLUTIONS FOR DIRICHLET PROBLEM II

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Abstract

It is proved that the space of solutions of the Dirichlet problem for the harmonic functions in the unit disk with nontangential boundary limits 0 a.e. has the infinite dimension.

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By the well-known Lindelöf maximum principle, see e.g. Lemma 1.1 in [3], it follows the uniqueness theorem for the Dirichlet problem in the class of bounded harmonic functions u on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation even under zero boundary data. In comparison with the previous arXiv versions and [7], here we give more elementary examples and constructions of solutions.

Many such nontrivial solutions u for the Laplace equation can be given by the **Poisson-Stieltjes integral**

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} P_r(\vartheta - t) d\Phi(t), \quad z = re^{i\vartheta}, \quad r < 1, \quad (1)$$

with an arbitrary **singular function** $\Phi : [0, 2\pi] \rightarrow \mathbb{R}$, i.e., where Φ is of bounded variation and $\Phi' = 0$ a.e., where we use the standard notation for the **Poisson kernel**

$$P_r(\Theta) = \frac{1 - r^2}{1 - 2r \cos \Theta + r^2}, \quad r < 1.$$

Indeed, u in (1) is harmonic for every function $\Phi : [0, 2\pi] \rightarrow \mathbb{R}$ of bounded variation and by the Fatou theorem, see e.g. Theorem I.D.3.1 in [6], $u(z) \rightarrow \Phi'(\Theta)$ as $z \rightarrow e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists. Thus, $u(z) \rightarrow 0$ as $z \rightarrow e^{i\Theta}$ for a.e. $\Theta \in [0, 2\pi]$ along any nontangential paths for every singular function Φ .

Example 1. The simplest example of such kind is given by nondecreasing step-like data Φ_{ϑ_0} with values 0 and 2π and with the jump at $\vartheta_0 \in (0, 2\pi)$:

$$u(z) = P_r(\vartheta - \vartheta_0) = \frac{1 - r^2}{1 - 2r \cos(\vartheta - \vartheta_0) + r^2}, \quad z = re^{i\vartheta}, \quad r < 1. \quad (2)$$

We directly see that $u(z) \rightarrow 0$ as $z \rightarrow e^{i\Theta}$ for all $\Theta \in (0, 2\pi)$ except $\Theta = \vartheta_0$.

Note that the function u is harmonic in the unit disk \mathbb{D} because

$$u(z) = \operatorname{Re} \frac{\zeta_0 + z}{\zeta_0 - z} = \frac{1 - |z|^2}{1 - 2 \operatorname{Re} z \overline{\zeta_0} + |z|^2}, \quad \zeta_0 = e^{i\vartheta_0}, \quad z \in \mathbb{D}, \quad (3)$$

where the function $w = g(z) = g_{\zeta_0}(z) := (\zeta_0 + z)/(\zeta_0 - z)$ is analytic (conformal) in \mathbb{D} and maps \mathbb{D} onto half-plane $\operatorname{Re} w > 0$, $g(0) = 1$, $g(\zeta_0) = \infty$.

Example 2. The second natural example is given by the formula (1) with $\Phi(t) = \varphi(t/2\pi)$ where $\varphi : [0, 1] \rightarrow [0, 1]$ is the well-known **Cantor function**, see e.g. [1] and further references therein.

The formula (2) gives a continual set of such examples. Furthermore, one can prove the following result.

Theorem 1. *The space of all harmonic functions in \mathbb{D} with nontangential limit 0 at every point of $\partial\mathbb{D}$ except a countable collection of points in $\partial\mathbb{D}$ has the infinite dimension.*

Proof. Indeed, let us consider the sequence of functions of the form (3) :

$$u_n(z) = \operatorname{Re} \frac{\zeta_n + z}{\zeta_n - z} = \frac{1 - |z|^2}{1 - 2 \operatorname{Re} z \overline{\zeta_n} + |z|^2}, \quad \zeta_n = e^{i\vartheta_n}, \quad z \in \mathbb{D},$$

where

$$\vartheta_n = \pi(2^{-1} + \dots + 2^{-n}), \quad n = 1, 2, \dots$$

and denote by \mathcal{H}_1 the class of all series $u = \sum \gamma_n u_n$ whose sequences of coefficients $\gamma = \{\gamma_n\}$ belong to the space l^1 with the norm $\|\gamma\| = \sum_{n=1}^{\infty} |\gamma_n| < \infty$.

Note that \mathcal{H}_1 consists of harmonic functions, see, e.g., Theorem I.3.1 in [5], because

$$0 < u_n(z) < \frac{1 + |z|}{1 - |z|} \quad \forall n = 1, 2, \dots, z \in \mathbb{D}.$$

Note also that each function $u \in \mathcal{H}_1$ has nontangential limit 0 at every point $\zeta \in \partial\mathbb{D}$ except the points $\zeta_0 = -1 = e^{i\vartheta_0}$, $\vartheta_0 = \pi$, and ζ_n , $n = 1, 2, \dots$. Indeed, let $\zeta = e^{i\Theta}$, $\Theta \in (0, 2\pi)$, $\zeta \neq \zeta_n$, $n = 0, 1, 2, \dots$. Then, applying the formula (2), we have the estimate

$$u_n(z) \leq \frac{1 - r^2}{4r \sin^2 \frac{\Theta - \vartheta_n}{4}} \leq C(1 - r), \quad z = re^{i\vartheta},$$

for all points z belonging to a sector $|\vartheta - \Theta| < c(1 - r)$ and for all r which are close enough to 1 where $C < \infty$ does not depend on $n = 1, 2, \dots$. Thus,

$$|u(z)| \leq C \|\gamma\| (1 - r) \rightarrow 0 \quad \text{as } r \rightarrow 1, \quad z = re^{i\vartheta},$$

in any sector $|\vartheta - \Theta| < c(1 - r)$.

Now, let us show that u_n , $n = 1, 2, \dots$, form a basis in the space \mathcal{H}_1 with the locally uniform convergence in \mathbb{D} which is metrizable.

Indeed, firstly, $u = \sum_{n=1}^{\infty} \gamma_n u_n \neq 0$ if $\gamma \neq 0$. Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \dots$. Then $u \neq 0$ because $u(z) \rightarrow \infty$ as $z = re^{i\vartheta_n} \rightarrow e^{i\vartheta_n}$. The latter follows because

$$u_n(re^{i\vartheta_n}) = \frac{1 + r}{1 - r} \rightarrow \infty \quad \text{as } r \rightarrow 1,$$

and by the previous item

$$|\tilde{u}(re^{i\vartheta_n})| \leq C \|\gamma\| (1 - r) \rightarrow 0 \quad \text{as } r \rightarrow 1,$$

where $\tilde{u} = u - \gamma_n u_n$.

Secondly, $u_m^* = \sum_{n=1}^m \gamma_n u_n \rightarrow u$ locally uniformly in \mathbb{D} as $m \rightarrow \infty$. Indeed, elementary calculations give the following estimate of the remainder term

$$|u(z) - u_m^*(z)| \leq \frac{1 + r}{1 - r} \cdot \sum_{n=m+1}^{\infty} |\gamma_n| \rightarrow 0 \quad \text{as } m \rightarrow \infty \quad (4)$$

in every disk $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \leq r\}$, $r < 1$. \square

Corollary 1. *Given a measurable function $\varphi : \partial\mathbb{D} \rightarrow \mathbb{R}$, the space of all harmonic functions $u : \mathbb{D} \rightarrow \mathbb{R}$ with the limits $\lim_{z \rightarrow \zeta} u(z) = \varphi(\zeta)$ for a.e. $\zeta \in \partial\mathbb{D}$ along nontangential paths has the infinite dimension.*

Indeed, the existence at least one such a harmonic function u follows from the known Gehring theorem in [4]. Combining this fact with Theorem 1, we obtain the conclusion of Corollary 1.

Remark 1. In view of Lemma 3.1 in [2], one can similarly prove the more refined result on harmonic functions than in Corollary 1 with respect to logarithmic capacity instead of the measure of the length on $\partial\mathbb{D}$.

Moreover, the statements on the infinite dimension of the space of solutions can be extended to the Riemann-Hilbert problem because the latter is reduced in the papers [2] and [7] to the corresponding two Dirichlet problems.

Note also that harmonic functions u found in Theorem 1 themselves cannot be represented in the form of the Poisson integral with any integrable function $\Phi : [0, 2\pi] \rightarrow \mathbb{R}$ because this integral would have nontangential limits Φ a.e., see e.g. Corollary IX.9.1 in [5]. Consequently, u do not belong to the classes h_p for any $p > 1$, see e.g. Theorem IX.2.3 in [5].

However, the functions $u \in \mathcal{H}_1$ in the proof of Theorem 1 have the representation as the Poisson-Stieltjes integral (1) with $\Phi = \sum \gamma_n \Phi_{\vartheta_n}$ where $\Phi_{\vartheta_n} : [0, 2\pi] \rightarrow \mathbb{R}$ are nondecreasing step-like functions with values 0 and 2π with jumps at the points ϑ_n , $n = 1, 2, \dots$. Thus, Φ is of bounded variation and hence $\mathcal{H}_1 \subset h_1$, see e.g. Theorem IX.2.2 in [5].

Problem 1. It remains the open question whether the basis of the space of all such singular solutions of the Dirichlet problem for the Laplace equation has the power of the continuum.

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